Categoricity for Patterns of Order 2

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In this paper we show how a Categoricity Theorem for patterns of resemblance of order 2, in analogy to Theorem 9.1 of [1] for $\mathcal{R}_1$, follows from [2]. This is the result alluded to in the last paragraph of the introduction to [2] where it is stated

> *... a method of generating the core is established which shows that the order in which patterns of embeddings of this level occur is the same for reasonable hierarchies.*

As a consequence, if a reasonable hierarchy $\mathcal{B}$ (see the Categoricity Theorem below) has arbitrary long finite chains in the interpretation of $\preceq_2$ then a finite structure is a pattern of resemblance of order two iff it is isomorphic to a finite substructure of $\mathcal{B}$ (see Corollary 0.8). These results apply to the version of $\mathcal{R}_2$ defined in the introduction to [2] as initial segments are reasonable hierarchies.

Our basic reference is [2].

We will work in the theory $\mathsf{KP}\omega$ i.e. Kripke-Platek Set Theory plus the Axiom of Infinity.

Fix a language $\mathcal{L}$ including the binary relation symbol $\preceq$. Let $\mathcal{L}_2$ be the expansion of $\mathcal{L}$ by binary relation symbols $\preceq_1$ and $\preceq_2$. We also write $\preceq_0$ for $\preceq$. We use *structure* to refer to what is more commonly called a partial structure where the interpretations of the function symbols are allowed to be partial. We will write $|\mathbf{P}|$ for the universe of a structure $\mathbf{P}$.

For the remainder of the paper, let $\mathcal{R}$ be an EM structure (see Section 3 of [2]) for $\mathcal{L}$ on the class of ordinals with $\preceq^{\mathcal{R}}$ the usual ordering. We assume the restriction of $\mathcal{R}$ to any ordinal is a set, there is a restriction with ω indecomposables and the indecomposables are cofinal in the ordinals.

Since $\mathcal{R}$ is an EM structure, it can be recovered by its restriction to the ω^{th} indecomposable which implies the set of indecomposables is Δ-definable and the function which maps an indecomposable λ to $\mathcal{R} \restriction \lambda$ is Σ-definable.

We also assume $\mathcal{B}$ is a structure for $\mathcal{L}_2$ whose arithmetic part (i.e. restriction to $\mathcal{L}$) is an arithmetic structure with respect to $\mathcal{R}$ (Definition 4.1 of [2]) in which the interpretation of each function symbol is total. We do not require that $\mathcal{B}$ be well-ordered with respect to the interpretation of $\preceq$ though our main focus will be on those $\mathcal{B}$ which are. Recall that $\preceq_k^{\mathcal{B}}$ respects $\preceq_{k-1}^{\mathcal{B}}$ if

$$\alpha \preceq_{k-1}^{\mathcal{B}} \beta \preceq_{k-1}^{\mathcal{B}} \gamma \text{ and } \alpha \preceq_k^{\mathcal{B}} \gamma \quad \Longrightarrow \quad \alpha \preceq_k^{\mathcal{B}} \beta$$

for all α, β, γ.

Categoricity Theorem for $\mathcal{R}_2$. *If*

(a) For $k = 1, 2$, $\mathcal{B} \restriction \alpha \preceq_k^{\infty} \mathcal{B} \restriction \beta$ whenever $\alpha \preceq_k^{\mathcal{B}} \beta$.

(b) $\preceq_1^{\mathcal{B}}$ and $\preceq_2^{\mathcal{B}}$ are partial orderings of the universe of $\mathcal{B}$ with $\preceq_2^{\mathcal{B}} \subseteq \preceq_1^{\mathcal{B}} \subseteq \preceq_0^{\mathcal{B}}$.

(c) $\preceq_k^{\mathcal{B}}$ respects $\preceq_{k-1}^{\mathcal{B}}$ for $k = 1, 2$.

(d) The arithmetic part of $\mathcal{B}$ is $\mathcal{R} \restriction \lambda$ for some λ which is indecomposable in $\mathcal{R}$.

then the core of $\mathcal{B}$ is isomorphic to an initial segment of the core of $\mathcal{R}_2 \restriction \lambda$.

$\mathcal{R}_2$ is defined in Definition 5.4 of [2].

For the rest of the paper, assume $\mathcal{B}$ satisfies (a)-(c) of the theorem. We do not assume that $\mathcal{B}$ is necessarily well-ordered by $\preceq^{\mathcal{B}}$.

Definition 0.1 *A pattern* $\mathbf{P}$ *is* $\mathcal{B}$**-covered** *if there is a covering of* $\mathbf{P}$ *in* $\mathcal{B}$.

See Definition 5.3 of [2] for the definition of *pattern* (short for *pattern of resemblance of order two*). See Definition 5.2 of [2] for the definition of *covering*. That definition is slightly different from that used in [1] in that the range of a covering is required to be closed (Definition 2.3 of [2]).

Definition 0.2 *Assume* $\mathbf{P}$ *is a pattern, h is a function from the universe of* $\mathbf{P}$ *into the universe of $\mathcal{B}$ and φ is a regressive function on the nonminimal indecomposable elements in the range of h (i.e. $h(\alpha) < \alpha$ for any nonminimal*

element in the range of h which is indecomposable in $\mathcal{R}$). Suppose also that $\mathbf{P}$ *is a closed substructure of the pattern* $\mathbf{P}^+$*. A function* h^+ *of the universe of* $\mathbf{P}^+$ *into the universe of* $\mathcal{B}$ **extends** h **above** φ *if* h^+ *extends* h *and*

$$\varphi(h(a)) < h^+(b)$$

for any indecomposable b *in* $\mathbf{P}^+$ *and any indecomposable* a *in* $\mathbf{P}$ *such that* $(-\infty, a)^{\mathbf{P}} \prec^{\mathbf{P}^+} b \prec^{\mathbf{P}^+} a$.

Definition 0.3 *Assume* $\mathbf{P}$ *and* $\mathbf{P}^+$ *are patterns and* $\mathbf{P}$ *is a closed substructure of* $\mathbf{P}^+$*. The rule* $\mathbf{P}|\mathbf{P}^+$ *is* **cofinally valid** *in* $\mathcal{B}$ *if for every covering* h *of* $\mathbf{P}$ *in* $\mathcal{B}$ *and every regressive function* φ *on the nonminimal indecomposable elements in the range of* h *there is a covering* h^+ *of* $\mathbf{P}^+$ *into* $\mathcal{B}$ *which extends* h *above* φ*.*

Lemma 0.4 *Every generating rule is cofinally valid in* $\mathcal{B}$*.*

Proof. The only properties of $\mathcal{R}_2$ used in the proof of part 2 of Lemma 13.11 of [2] and the supporting lemmas are the preliminary properties we have assumed of $\mathcal{B}$. Therefore, the proof carries over with $\mathcal{R}_2$ replaced by $\mathcal{B}$.

The proof is by induction on the generation of the generating rules (Definition 13.10 of [2]).

Suppose $\mathbf{P}^+$ is 1-correct arithmetic extension of $\mathbf{P}$. The proof that $\mathbf{P}|\mathbf{P}^+$ is cofinally valid in $\mathcal{B}$ is analogous to the proof of Lemma 8.4 of [2]. Assume h is a covering of $\mathbf{P}$ in $\mathcal{B}$ and φ is a regressive function on the nonminimal indecomposables in the range of h. Notice that any covering of $\mathbf{P}^+$ in $\mathcal{B}$ which extends h vacuously extends h above φ since there are no new indecomposable elements (by Lemma 4.9 of [2]). By Lemma 4.5 of [2], there is an embedding h^+ of the arithmetic part of $\mathbf{P}^+$ in $\mathcal{R}$ which extends h. Clearly, the range of h^+ is contained in λ. A straightforward argument using the fact that $\mathbf{P}^+$ is a 1-correct arithmetic extension of $\mathbf{P}$ (Definitions 4.8, 7.1 and 8.1 of [2]) shows h^+ is a covering of $\mathbf{P}^+$ in $\mathcal{B}$.

Suppose $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting X downward from b to a. The proof that $\mathbf{P}|\mathbf{P}^+$ is cofinally valid in $\mathcal{B}$ is analogous to the proof of Lemma 9.3 of [2]. Assume h is a covering of $\mathbf{P}$ in $\mathcal{B}$ and φ is a regressive function on the nonminimal indecomposables in the range of h. Since h is a covering and $a \prec_1^{\mathbf{P}} b$, $h(a) \prec_1^{\mathcal{B}} h(b)$ implying $h(a) \prec_1^{\infty} h(b)$ in $\mathcal{B}$. Therefore, there is $\tilde{X}$ such that $h[(-\infty, a)^{\mathbf{P}}] \cup \{\varphi(h(a))\} < \tilde{X} < h(b)$ and $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{X}$ is both closed and a covering of $h[(-\infty, a)^{\mathbf{P}}] \cup h[X]$. Let h^+ be the order

isomorphism of $\mathbf{P}^+$ and $h[|\mathbf{P}|] \cup \tilde{X}$. A straightforward argument using the fact that $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 1-reflecting X downward from a to b (Definition 9.1 of [2]) shows that h^+ is a covering of $\mathbf{P}^+$ which extends h above φ.

Suppose $\mathbf{P}^+$ is obtained from $\mathbf{P}$ by 2-reflecting X downward from b to a. The proof that $\mathbf{P}|\mathbf{P}^+$ is cofinally valid in $\mathcal{B}$ is analogous to the proof of Lemma 9.6 of [2] and similar to the proof in the previous paragraph (using Definition 9.4 of [2] instead of Definition 9.1).

Assume $\mathbf{P}|\mathbf{P}^+$ is a generating rule which is cofinally valid in $\mathcal{B}$ and $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from a to b. The proof that $\mathbf{P}|\mathbf{P}^*$ is cofinally valid in $\mathcal{B}$ is analogous to the proof of Lemma 10.3 of [2]. Let $X = |\mathbf{P}^+| \setminus |\mathbf{P}|$. By Definition 10.1 of [2], $\mathbf{P}^+$ is a continuous extension of $\mathbf{P}$ at a (see Definitions 7.1 and 7.4 of [2]) and $a \preceq_2^{\mathbf{P}} b$. Assume h is a covering of $\mathbf{P}$ in $\mathcal{B}$ and φ is a regressive function on the nonminimal indecomposables in the range of h. Since $a \preceq_2^{\mathbf{P}} b$, $h(a) \preceq_2^{\mathcal{B}} h(b)$ implying $h(a) \preceq_2^{\infty} h(b)$ in $\mathcal{B}$. Since $\mathbf{P}|\mathbf{P}^+$ is cofinally valid in $\mathcal{B}$, there are cofinally many $\tilde{X}$ below $h(a)$ such that $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{X}$ is closed and a covering of $(-\infty, a)^{\mathbf{P}} \cup X$ (as a substructure of $\mathbf{P}^+$). Since $h(a) \preceq_2^{\infty} h(b)$ in $\mathcal{B}$, there are cofinally many $\tilde{X}$ below $h(b)$ such that $h[(-\infty, a)^{\mathbf{P}}] \cup \tilde{X}$ is closed and a covering of $(-\infty, a)^{\mathbf{P}} \cup X$. Choose such $\tilde{X}$ such that $\varphi(h(b)) < \tilde{X}$. A straightforward argument using the fact that $\mathbf{P}|\mathbf{P}^*$ is obtained by 2-reflecting $\mathbf{P}|\mathbf{P}^+$ upward from a to b (Definition 10.1 of [2]) shows that h^+ is a covering of $\mathbf{P}^*$ which extends h above φ.

Assume $\mathbf{P}_i|\mathbf{P}_{i+1}$ is a generating rule which is cofinally valid in $\mathcal{B}$ for $i < n$ and $\mathbf{P}^+$ is a closed substructure of $\mathbf{P}_n$ which extends $\mathbf{P}_0$. An easy argument by induction shows $\mathbf{P}_0|\mathbf{P}_i$ is cofinally valid in $\mathcal{B}$ for $i \leq n$. The fact that $\mathbf{P}_0|\mathbf{P}_n$ is cofinally valid in $\mathcal{B}$ clearly implies that $\mathbf{P}_0|\mathbf{P}^+$ is also.

Assume $\mathbf{P}|\mathbf{P}^+$ is a generating rule which is cofinally valid in $\mathcal{B}$ and h is a continuous embedding of $\mathbf{P}$ in $\mathbf{Q}$. Let $\mathbf{Q}^+$ be a minimal lifting (Definitions 12.1 and 12.4 of [2]) of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$ with respect to h and let h^+ be the lifting map. The proof that $\mathbf{Q}|\mathbf{Q}^+$ is cofinally valid in $\mathcal{B}$ is analogous to the proof of Lemma 13.8 of [2]. By identifying $\mathbf{P}$ and $\mathbf{P}^+$ with their images under h^+, we may assume h^+ is the identity on $|\mathbf{P}^+|$. Assume f is a covering of $\mathbf{Q}$ in $\mathcal{B}$ and assume φ is a regressive function on the nonminimal indecomposables in the range of f. By increasing the values of φ if necessary, we may assume that $f[(-\infty, a)^{\mathbf{Q}}] \leq \varphi(h(a))$ whenever $a \in |\mathbf{P}|$ and $h(a)$ is indecomposable. Since $\mathbf{P}|\mathbf{P}^+$ is cofinally valid in $\mathcal{B}$, there is a covering g of $\mathbf{P}^+$ in $\mathcal{B}$ which extends the restriction of f to $|\mathbf{P}|$ above the restriction of φ to the indecomposables in $f[|\mathbf{P}|]$. The restriction of $f \cup g$ to the idecomposables of $\mathbf{Q}^+$ is order

preserving. By Lemma 4.5 of [2], this map extends to a unique arithmetic embedding of the arithmetic part of $\mathbf{Q}^+$ in $\mathcal{B}$ which must extend both f and g. Therefore, $f \cup g$ is an arithmetic embedding of the arithmetic part of $\mathbf{Q}^+$ in $\mathcal{B}$. Let $\mathbf{Q}^*$ be the pattern with the same arithmetic part as $\mathbf{Q}^+$ which is induced by $\mathcal{B}$ through $f \cup g$ i.e. so that $f \cup g$ is an embedding of $\mathbf{Q}^*$ in $\mathcal{B}$. Consider the structure $\mathbf{Q}'$ which has the same arithmetic part as $\mathbf{Q}^+$ so that the interpretation of $\preceq_k$ is the intersection of the interpretations of $\preceq_k$ in $\mathbf{Q}^+$ and $\mathbf{Q}^*$. A straightforward argument shows $\mathbf{Q}'$ is a lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$. Since $\mathbf{Q}^+$ is a minimal lifting of $\mathbf{P}|\mathbf{P}^+$ to $\mathbf{Q}$, $\mathbf{Q}'$ must be a cover of $\mathbf{Q}^+$ (actually, equal to $\mathbf{Q}^+$) implying $\mathbf{Q}^*$ is a cover of $\mathbf{Q}^+$. Therefore, $f \cup g$ is a covering of $\mathbf{Q}^+$ in $\mathcal{B}$. Clearly, $f \cup g$ extends f above φ. **QED**

Lemma 0.5 *Assume* $\mathbf{P}$ *and* $\mathbf{Q}$ *are patterns and* $\mathbf{P}$ *generates* $\mathbf{Q}$*. Any covering of* $\mathbf{P}$ *in* $\mathcal{B}$ *extends to a covering of* $\mathbf{Q}$ *in* $\mathcal{B}$*.*

Proof. Straightforward from the previous lemma (see Definition 14.2 of [2]). **QED**

The following two lemmas will be used only to show that if the arithmetic part of $\mathcal{B}$ is the restriction of $\mathcal{R}$ to an indecomposable of $\mathcal{R}$ then every $\mathcal{B}$-covered pattern is covered i.e. $\mathcal{R}_2$-covered. Hence, if one is willing to accept the assumption that every pattern is covered (which increases the proof-theoretic strength of the metatheory to just beyond $\mathsf{KP}\ell_0$) then these lemmas can be omitted.

The next lemma is an observation that the proofs of parts 3, 4, 6 and 8 of Lemma 14.8 in [2] actually prove stronger assertions. Notice that in our base theory $\mathsf{KP}\omega$, saying that a linear ordering is order isomorphic to an ordinal is stronger than saying it is a well-ordering.

Lemma 0.6 *Assume* $\mathbf{P}_n$ $(n \in \omega)$ *is an increasing sequence of patterns such that* $\mathbf{P}_n|\mathbf{P}_{n+1}$ *is a generating rule for each* $n \in \omega$*. Let* $\mathbf{P}_\infty$ *be the union of the* $\mathbf{P}_n$ $(n \in \omega)$*.*

- *3*. Every covering of* $\mathbf{P}_0$ *in* $\mathcal{B}$ *extends to a covering of* $\mathbf{P}_\infty$ *in* $\mathcal{B}$*.*
- *4*. Assume* $(|\mathcal{B}|, \preceq^{\mathcal{B}})$ *is order isomorphic to an ordinal. If* $\mathbf{P}_0$ *is* $\mathcal{B}$*-covered then* $(|\mathbf{P}_\infty|, \preceq^{\mathbf{P}_\infty})$ *is order isomorphic to an ordinal*
- *6* If* $\mathbf{P}_\infty$ *is a well-ordered structure (i.e.* $\preceq^{\mathbf{P}_\infty}$ *is a well-ordering of* $\mathbf{P}_\infty$*) and* $\mathbf{Q}$ *is a closed substructure of* $\mathbf{P}_\infty$ *which is a covering of* $\mathbf{P}_n$ *then* $|\mathbf{P}_n| \preceq_{pw}^{\mathbf{P}_\infty} |\mathbf{Q}|$*.*

8. Assume $\mathbf{P}_n$ ($n \in \omega$) is fair and $\mathbf{P}_\infty$ is a well-ordered structure.*

(a) For $k = 1, 2$ and $a, b \in |\mathcal{R}|$

$$a \preceq_k^\infty b \quad \Longrightarrow \quad a \preceq_k^{\mathbf{P}_\infty} b$$

(b) If $(|\mathbf{P}_\infty|, \preceq^{\mathbf{P}_\infty})$ is order isomorphic to an ordinal then $\mathbf{P}_\infty$ is isomorphic to $\mathcal{R}_2 \restriction \delta$ for some δ which is indecomposable in $\mathcal{R}$.

Proof. Part 3* follows from Lemma 0.5.

Part 4* follows from part 3*.

For part 6*, notice that parts 1, 5 and 7 of Lemma 14.8 of [2] implies that $\mathbf{P}_\infty$ satisfies our preliminary assumptions on $\mathcal{B}$ i.e. the arithmetic part of $\mathbf{P}_\infty$ is an arithmetic structure with respect to $\mathcal{R}$ and parts (a)-(c) of the Categoricity Theorem hold. Taking $\mathcal{B}$ to be $\mathbf{P}_\infty$ in part 3* we see there is a covering of $\mathbf{P}_\infty$ into itself which extends the covering of $\mathbf{P}_n$ onto $\mathbf{Q}$. Since $\mathbf{P}_\infty$ is well-ordered, we must have $|\mathbf{P}_n| \preceq_{pw}^{\mathbf{P}_\infty} |\mathbf{Q}|$.

The proof of part 8 of Lemma 14.8 of [2] actually shows part 8*(a) if we replace applications of part 6 of Lemma 14.8 by applications of part 6* above.

For part 8*(b), we may assume the arithmetic part of $\mathcal{B}$ is $\mathcal{R} \restriction \delta$ for some ordinal δ which is indecomposable in $\mathcal{R}$ by parts 1 and 5 of Lemma 14.8 and Lemmas 4.4 and 4.5 of [2]. A simple induction using part 7 of Lemma 14.8 of [2] and part 8*(a) shows that for $\alpha \leq \delta$, the restriction of $\preceq_k^{\mathcal{B}}$ to α is the same as the restriction of $\preceq_k^{\mathcal{R}_2}$ to α for $k = 1, 2$. **QED**

Lemma 0.7 *If the arithmetic part of $\mathcal{B}$ is isomorphic to an initial segment of $\mathcal{R}$ then any $\mathcal{B}$-covered pattern is covered.*

Proof. Assume h is a covering of the pattern $\mathbf{P}$ in $\mathcal{B}$. Let $\mathbf{P}_n$ ($n \in \omega$) be a fair sequence of patterns with $\mathbf{P}_0 = \mathbf{P}$.

By part 3* of the previous lemma, there is a covering h^+ of $\mathbf{P}_\infty$ in $\mathcal{B}$ which extends h. By part 8*(b) of the previous lemma, $\mathbf{P}_\infty$ is isomorphic to an initial segment of $\mathcal{R}_2$. The restriction of that isomorphism to $|\mathbf{P}|$ is a covering of $\mathbf{P}$ in $\mathcal{R}_2$. **QED**

Proof of the Categoricity Theorem. Our proof will follow the general lines of the proof of Theorem 9.1 of [1].

Claim1. Assume $\mathbf{P}$ is $\mathcal{B}$-covered and $\mathbf{P}'$ is a minimal element with respect to $\preceq^{\mathcal{B}}_{pw}$ (the pointwise partial ordering of finite subsets of $\mathcal{B}$) among the closed substructures of $\mathcal{B}$ which are coverings of $\mathbf{P}$.

(i) If $\mathbf{Q}$ is a substructure of $\mathcal{B}$ which is a cover of $\mathbf{P}$ then $|\mathbf{P}'|\leq_{pw}|\mathbf{Q}|$.

(ii) $\mathbf{P} \cong \mathbf{P}'$.

For (i), suppose $\mathbf{Q}$ is a substructure of $\mathcal{B}$ which is a cover of $\mathbf{P}$. By Theorem 14.10 of [2], there are finite closed substructures $\mathbf{R}$ and $\mathbf{P}^*$ of $\mathcal{R}_2$ which are isominimal in $\mathcal{R}_2$ and isomorphic to $\mathbf{P}' \cup \mathbf{Q}$ (with a slight abuse of notation) and $\mathbf{P}$ respectively. Let $\overline{\mathbf{P}'}$ and $\overline{\mathbf{Q}}$ be the images of $\mathbf{P}'$ and $\mathbf{Q}$ respectively under the isomorphism of $\mathbf{P}' \cup \mathbf{Q}$ and $\mathbf{R}$. By part 2 of Theorem 14.10 of [2], $|\mathbf{P}^*|\leq_{pw}|\overline{\mathbf{P}'}|, |\overline{\mathbf{Q}}|$. By part 5 of Theorem 14.10 of [2], $\overline{\mathbf{P}'} \cup \overline{\mathbf{Q}}$ generates $\mathbf{P}^* \cup \overline{\mathbf{P}'} \cup \overline{\mathbf{Q}}$. By Lemma 0.5, there is a covering h of $\mathbf{P}^* \cup \overline{\mathbf{P}'} \cup \overline{\mathbf{Q}}$ in $\mathcal{B}$ which extends the isomorphism of $\overline{\mathbf{P}'} \cup \overline{\mathbf{Q}}$ with $\mathbf{P}' \cup \mathbf{Q}$. Let $\mathbf{P}''$ be the image of $\mathbf{P}^*$ under h. We have $|\mathbf{P}''|\leq_{pw}|\mathbf{P}'|, |\mathbf{Q}|$. By the minimality of $\mathbf{P}'$, $\mathbf{P}'' = \mathbf{P}'$. Therefore, $|\mathbf{P}'|\leq_{pw}|\mathbf{Q}|$.

For part (ii), follow the argument for part (i) (one may take $\mathbf{Q} = \mathbf{P}'$) to conclude from $\mathbf{P}'' = \mathbf{P}'$ that $\mathbf{P}^* = \overline{\mathbf{P}'}$. Since $\mathbf{P}^* \cong \mathbf{P}$ and $\overline{\mathbf{P}'} \cong \mathbf{P}'$, $\mathbf{P} \cong \mathbf{P}'$.

For any covered pattern $\mathbf{P}$, let $\mathbf{P}^*$ be the isominimal substructure of $\mathcal{R}_2$ which is isomorphic to $\mathbf{P}$. For $\mathbf{P}$ an isominimal substructure of $\mathcal{B}$, define $f_{\mathbf{P}}$ to be the isomorphism of $\mathbf{P}$ and $\mathbf{P}^*$. Let f be the union of the $f_{\mathbf{P}}$. A straightforward argument shows f is an embedding of the core of $\mathcal{B}$ into the core of $\mathcal{R}_2$.

To show the range of f is an initial segment of $\mathcal{R}_2$, assume $\alpha < \beta$ where β is in the range of f. There is an isominimal substructure $\mathbf{P}$ of $\mathcal{B}$ such that β is in the range of $f_{\mathbf{P}}$. Let $\mathbf{P}_n$ $(n \in \omega)$ be a fair sequence with $\mathbf{P}_0 = \mathbf{P}$ and let $\mathbf{P}_\infty$ be the union of the $\mathbf{P}_n$. By Lemma 14.9 of [2], there is an isomophism g of $\mathbf{P}_\infty$ with $\mathcal{R}_2 \restriction \delta$ for some δ which is indecomposable in $\mathcal{R}$ and the image of $\mathbf{P}_n$ under g is $\mathbf{P}^*_n$ for each $n \in \omega$. Fix n such that α is in $\mathbf{P}^*_n$. By Lemma 0.5 and Claim 1, there is an isominimal substructure $\mathbf{Q}$ of $\mathcal{B}$ which is isomorphic to $\mathbf{P}_n$. Since α is in $\mathbf{P}^*_n$ which the range of $f_{\mathbf{Q}}$, α is in the range of f. **QED**

Corollary 0.8 *Assume $\mathcal{B}$ satisfies (a)-(d) of the Categoricity Theorem for $\mathcal{R}_2$. If there are arbitrarily long finite chains in $\preceq^{\mathcal{B}}_2$ then the core of $\mathcal{B}$ is isomorphic to the core of $\mathcal{R}_2$ and a finite structure is isomorphic to a finite closed substructure of $\mathcal{B}$ iff it is a pattern of resemblance of order 2.*

Proof. Assume there are arbitrarily long finite chains in $\preceq_2^{\mathcal{B}}$. By the Categoricity Theorem, the core of $\mathcal{B}$ is isomorphic to an initial segment of the core of $\mathcal{R}_2$. Since this initial segment contains arbitrarily long finite chains in $\leq_2$, it must be the entire core of $\mathcal{R}_2$ by part 2 of Theorem 14.10 of [2]. Hence, every pattern of resemblance of order two is isomorphic to a substructure of $\mathcal{B}$. The converse is straightforward after noticing that condition (a) of the Categoricity Theorem implies that α is indecomposable whenever $\alpha \preceq_1^{\mathcal{B}} \beta$ and both α and β are indecomposable whenever $\alpha \preceq_2^{\mathcal{B}} \beta$. **QED**

Corollary 0.9 *Assume $\mathcal{R}_2'$ is the alternate definition of $\mathcal{R}_2$ from the introduction to* [2] *using Σ_1 and Σ_2 elementarity. $\mathcal{R}_2' \restriction \delta$ satisfies the (a)-(d) of the Categoricity Theorem for each indecomposable δ and, hence, the conclusions of the Categoricity Theorem and the previous corollary hold for $\mathcal{R}_2$.*

Proof. Straightforward after noting that in $\mathcal{R}_2'$, if $\alpha < \beta$, β is a limit ordinal and $\alpha \leq_1 \xi$ for all ξ with $\alpha \leq \xi < \beta$ then $\alpha \leq_1 \beta$. **QED**

One can prove that $\leq_2$ in $\mathcal{R}_2'$ has arbitrarily long finite chains well within ZF.

References

1. *Elementary patterns of resemblance,* Annals of Pure and Applied Logic 108 (2001), pp. 19-77.

2. *Patterns of resemblance of order 2,* Annals of Pure and Applied Logic 158 (2009), pp. 90-124.